AN EXISTENCE THEOREM FOR FIELDS WITH KRULL VALUATIONS

BY

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Introduction. (a) In 1925, Hasse [2] proved the following existence theorem:

Let $\boldsymbol{K}$ be a field of algebraic numbers, and $\boldsymbol{P}_1,\dots,\boldsymbol{P}_s$ prime ideals of the ring $\boldsymbol{A}$ of algebraic integers of $\boldsymbol{K}$; for each $i = 1,\dots,s$ denote by $\bar{\boldsymbol{K}}_i = \boldsymbol{A}/\boldsymbol{P}_i$ the residue class field of $\boldsymbol{A}$ modulo $\boldsymbol{P}_i$. Suppose given, for each $i = 1,\dots,s$, g_i positive integers $e_{i1},\dots,e_{ig_i}$ and g_i (non necessarily distinct) algebraic extensions $\bar{\boldsymbol{K}}_{ij} \mid \bar{\boldsymbol{K}}_i$ of finite degree f_{ij} $(j = 1,\dots,g_i)$ in such a way that

$$\sum_{j=1}^{g_i} e_j \cdot f_{ij} = n \qquad \text{(for every } i = 1,\dots,s). \tag{1}$$

Then, there exists a finite extension K of $\boldsymbol{K}$, having degree n, such that, for each $i = 1,\dots,s$, the prime ideal $\boldsymbol{P}_i$ decomposes in the field K as a product

$$\boldsymbol{P}_i = \prod_{j=1}^{g_i} P_{ij}^{e_{ij}},$$

P_{ij} being prime ideals of the ring A of algebraic integers of K, having residue class field $A/P_{ij} = \bar{K}_{ij}$.

(b) In 1959, Krull [5] indicated (among other results) the following generalization of Hasse's theorem:

Let $\boldsymbol{K}$ be a field, let $\boldsymbol{v}_1,\dots,\boldsymbol{v}_s$ be discrete valuations (of rank 1) of $\boldsymbol{K}$, and assume that $\boldsymbol{K}$ has at least one further discrete valuation $\boldsymbol{v}$; for each $i = 1,\dots,s$, denote by $\bar{\boldsymbol{K}}_i = \boldsymbol{K}/\boldsymbol{v}_i$ the residue class field of $\boldsymbol{K}$ with respect to the valuation $\boldsymbol{v}_i$.

Suppose given, for each $i = 1,\dots,s$, g_i positive integers $e_{i1},\dots,e_{ig_i}$ and g_i (not necessarily distinct) simple extensions $\bar{K}_{ij}$ of $\bar{\boldsymbol{K}}_i$ of finite degree f_{ij} $(j = 1,\dots,g_i)$ [in particular, this is the case when $\bar{K}_{ij} \mid \boldsymbol{K}_i$ are finite separable extensions, for example, when the fields $\bar{\boldsymbol{K}}_i$ are perfect]; furthermore, we assume that for every $i = 1,\dots,s$ the relation (1) holds. Then, there exists a finite separable extension K of $\boldsymbol{K}$, having degree n, such that for every $i = 1,\dots,s$, the valuation $\boldsymbol{v}_i$ of $\boldsymbol{K}$ has g_i distinct prolongations $v_{i1},\dots,v_{ig_i}$ to K, having ramification indices e_{ij} and residue class fields $K/v_{ij} = \bar{K}_{ij}$ $(j = 1,\dots,g_i)$.

Krull's theorem applies to fields of algebraic functions of one variable over a finite constant field, or over a constant field of characteristic zero; the extra valuation required in the hypothesis may be taken to be the valuation at the "point at infinity" in the classical sense.

(c) In our paper, cf. [8], we have proved a result along the line of Krull's

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theorem; instead of s valuations of K we dealt there with only *one* valuation, which was supposed to be *discrete of rank* $r \geqq 1$.

In the prolongation of this valuation to a separable extension, it is possible to specify independently the ramification indices, the extensions of the residue class field and the tree to be generated by the prolongations of the valuation.

(d) It is our purpose here to complete our previous investigations, obtaining a similar theorem, starting from a finite set of discrete valuations of rank $r \geqq 1$ in a field $\boldsymbol{K}$, which may generate any tree whatsoever.

We obtain an existence theorem, whose full statement will be given after adequate terminology is introduced.

1. **Definitions.** In order to prepare for the statement of the main theorem we must introduce several concepts.

DEFINITION. A *tree* (of *length* $r \geqq 0$) is a set $\mathfrak{T}$ satisfying:

(1) $\mathfrak{T}$ is a finite ordered set (by a relation $\leqq$);

(2) $\mathfrak{T}$ has first element δ;

(3) for every element $\gamma \in \mathfrak{T}$ the set $\{\gamma' \in \mathfrak{T} \mid \gamma' \leqq \gamma\}$ is totally ordered;

(4) all the maximal chains in $\mathfrak{T}$ have $r+1$ elements (these conditions are mutually independent).

It follows that:

(5) $\mathfrak{T}$ is an inf-lattice: if $\gamma, \gamma' \in \mathfrak{T}$ then there exists the infimum $\gamma \wedge \gamma' \in \mathfrak{T}$.

DEFINITION. Let $\mathfrak{T}$ be a tree. The *length of* $\gamma \in \mathfrak{T}$ is said to be $l_{\mathfrak{T}}(\gamma) = l$ whenever the set $\{\gamma' \in \mathfrak{T} \mid \gamma' \leqq \gamma\}$ has $l+1$ elements.

In particular, δ has length 0, the maximal elements of $\mathfrak{T}$ have length r (equal to the length of $\mathfrak{T}$).

We denote by $\mathfrak{M}$ the set of all maximal elements of $\mathfrak{T}$ and by $\mathfrak{P}$ the set of all elements of $\mathfrak{T}$ with length 1.

DEFINITION. A *value function of the tree* $\mathfrak{T}$ (of length r) is a mapping Γ that associates to each $\gamma \in \mathfrak{T}$ a totally ordered abelian additive group, $\Gamma(\gamma)$, satisfying the following conditions:

(6) $\Gamma(\gamma)$ is a group of rank r if and only if γ is a maximal element of $\mathfrak{T}$;

(7) if $\gamma' \leqq \gamma$ there exists a homomorphism (of ordered groups) $\theta_{\gamma,\gamma'} : \Gamma(\gamma) \to \Gamma(\gamma')$ which is onto $\Gamma(\gamma')$, and moreover, this homomorphism is an isomorphism if and only if $\gamma = \gamma'$;

(8) if $\gamma'' \leqq \gamma' \leqq \gamma$ then $\theta_{\gamma,\gamma''} = \theta_{\gamma',\gamma''} \circ \theta_{\gamma,\gamma'}$.

From the above conditions 6, 7, 8, we deduce:

(9) $\Gamma(\gamma)$ is a group of rank l if and only if γ is an element of length l of $\mathfrak{T}$;

(10) $\Gamma(\delta) = 0$, trivial group.

DEFINITION. A *field function of the tree* $\mathfrak{T}$ (of length r) is a mapping $\mathfrak{K}$ that associates to each $\gamma \in \mathfrak{T}$, $\gamma \neq \delta$, a field $\mathfrak{K}(\gamma)$, satisfying the following conditions:

(11) if $\gamma' \leqq \gamma$, $\gamma' \neq \delta$, there exists a place π_γ/γ' of $\mathfrak{K}(\gamma')$, with value field equal to $\mathfrak{K}(\gamma)$ (in particular, this place is trivial if and only if $\gamma' = \gamma$);

(12) if $\gamma'' \leqq \gamma' \leqq \gamma$, $\gamma'' \neq \delta$ then $\pi_\gamma/\gamma'' = (\pi_\gamma/\gamma') \circ (\pi_{\gamma'}/\gamma'')$.

Definition. Let $\mathfrak{T}$ be a tree, Γ a value function of $\mathfrak{T}$ and $\mathfrak{K}$ a field function of $\mathfrak{T}$. The triple $\mathfrak{C} = \{\mathfrak{T}, \Gamma, \mathfrak{K}\}$ is called a *configuration*. We say that $\mathfrak{C}$ has length equal to the length of the tree $\mathfrak{T}$.

Definition. Let $\mathfrak{C} = \{\mathfrak{T}, \Gamma, \mathfrak{K}\}$, $\mathfrak{C}' = \{\mathfrak{T}', \Gamma', \mathfrak{K}'\}$ be configurations. A mapping $I : \mathfrak{T} \to \mathfrak{T}'$ which is an order preserving isomorphism from $\mathfrak{T}$ onto $\mathfrak{T}'$ is called an *isomorphism of* $\mathfrak{C}$ *onto* $\mathfrak{C}'$ whenever the following conditions are satisfied:

(13) for every $\gamma \in \mathfrak{T} : \Gamma(\gamma) = \Gamma'(I(\gamma))$;

(14) for every $\gamma \in \mathfrak{T}$, $\gamma \neq \delta : \mathfrak{K}(\gamma) = \mathfrak{K}'(I(\gamma))$.

A finite set $\mathfrak{V}$ of valuations of finite rank r of a field K defines a configuration in the following way.

Let $\mathfrak{T}$ be the set of all valuations of K coarser than at least one of the given valuations. Then, $\mathfrak{T}$ is a tree of length r, as it is well known (cf. [6; 8]).

For each valuation $v \in \mathfrak{T}$ let us consider its value group $v(K)$; the mapping $v \to v(K)$ is a value function of the tree $\mathfrak{T}$. If $v' \leqq v$ in $\mathfrak{T}$, then there exists a prime ideal P of the ring of v, such that $v' = v_P$; if Δ is the isolated subgroup of $v(K)$ corresponding to P, we take $\theta_{v,v'} : v(K) \to v'(K)$ to be the quotient mapping by Δ.

For each nontrivial valuation $v \in \mathfrak{T}$, let us consider the residue class field K/v of K with respect to v; the mapping $v \to K/v$ is a field function of the tree $\mathfrak{T}$. If $v' \leqq v$ in $\mathfrak{T}$, if $v' = v_P$, let v/P be the valuation of K/v' having ring A/P; we take π_v/v' to be the place of the field K/v' associated with the valuation v/P.

Hence, we have assigned to K and the set $\mathfrak{V}$ of valuations the configuration $\mathfrak{C} = \{\mathfrak{T}, v \to v(K), v \to K/v\}$, which we call the *configuration of* K *generated by the set* $\mathfrak{V}$ of valuations of rank r.

Definition. A tree $\mathfrak{T}$ is a *prolongation of a tree* $\boldsymbol{\mathfrak{T}}$ (or lies over $\boldsymbol{\mathfrak{T}}$) when there exists a mapping $\rho : \mathfrak{T} \to \boldsymbol{\mathfrak{T}}$ such that:

(15) ρ is an order-preserving mapping onto $\boldsymbol{\mathfrak{T}}$;

(16) for every $\gamma \in \mathfrak{T}$: $l_{\boldsymbol{\mathfrak{T}}}(\gamma) = l_{\boldsymbol{\mathfrak{T}}}(\rho(\gamma))$;

(17) if γ, $\gamma' \in \mathfrak{T}$ and $\rho(\gamma) < \rho(\gamma')$ there exists $\gamma_1' \in \mathfrak{T}$ such that $\gamma \leqq \gamma_1'$ and $\rho(\gamma_1') = \rho(\gamma')$; (these conditions are mutually independent).

It follows that:

(18) if $\gamma < \gamma'$ in $\mathfrak{T}$ then $\rho(\gamma) < \rho(\gamma')$ in $\boldsymbol{\mathfrak{T}}$;

(19) $\rho(\gamma) = \boldsymbol{\delta}$ if and only if $\gamma = \delta$;

(20) $\rho(\gamma)$ is maximal in $\boldsymbol{\mathfrak{T}}$ if and only if γ is maximal in $\mathfrak{T}$;

(21) the length of $\mathfrak{C}$ is equal to the length of $\boldsymbol{\mathfrak{C}}$.

Definition. A configuration $\mathfrak{C} = \{\mathfrak{T}, \Gamma, \mathfrak{K}\}$ is a *prolongation of a configuration* $\boldsymbol{\mathfrak{C}} = \{\boldsymbol{\mathfrak{T}}, \boldsymbol{\Gamma}, \boldsymbol{\mathfrak{K}}\}$ (or lies over $\boldsymbol{\mathfrak{C}}$) whenever the tree $\mathfrak{T}$ is a prolongation of $\boldsymbol{\mathfrak{T}}$ (with a mapping ρ) in such a way that:

(22) for every $\gamma \in \mathfrak{T}$ the group $\boldsymbol{\Gamma}(\rho(\gamma))$ is an ordered subgroup of the group $\Gamma(\gamma)$;

(23) if $\gamma' \leqq \gamma$ then $\boldsymbol{\theta}_{\rho(\gamma),\rho(\gamma')}$ is the restriction of $\theta_{\gamma,\gamma'}$ to $\boldsymbol{\Gamma}(\rho(\gamma))$;

(24) for every $\gamma \in \mathfrak{T}$, $\gamma \neq \delta$, the field $\boldsymbol{\mathfrak{K}}(\rho(\gamma))$ is a subfield of $\mathfrak{K}(\gamma)$;

(25) if $\gamma' \leqq \gamma$, $\gamma' \neq \delta$, then $\boldsymbol{\pi}_{\rho(\gamma)}/\rho(\gamma')$ is the restriction of π_γ/γ' to $\boldsymbol{\mathfrak{K}}(\rho(\gamma'))$.

It follows that the groups $\Gamma(\gamma)$, $\boldsymbol{\Gamma}(\rho(\gamma))$ have the same rank, equal to $l_{\mathfrak{T}}(\gamma) = l_{\boldsymbol{\mathfrak{T}}}(\rho(\gamma))$.

The mapping ρ is called the *covering mapping* of $\mathfrak{C}$ over $\boldsymbol{\mathfrak{C}}$.

It is clear that if $\boldsymbol{\mathfrak{C}}$, $\mathfrak{C}$, $\mathfrak{C}'$ are configurations and $\mathfrak{C}$ lies over $\boldsymbol{\mathfrak{C}}$, $\mathfrak{C}'$ lies over $\mathfrak{C}$, then $\mathfrak{C}'$ lies over $\boldsymbol{\mathfrak{C}}$.

Let $\boldsymbol{K}$ be a field and $\boldsymbol{\mathfrak{B}}$ a finite set of valuations of rank r of $\boldsymbol{K}$; let K be an algebraic extension of degree n over $\boldsymbol{K}$ and $\mathfrak{B}$ be the (finite) set of valuations of K extending the valuations in $\boldsymbol{\mathfrak{B}}$.

Then, the configuration $\mathfrak{C}$ of K, generated by the set $\mathfrak{B}$, is a prolongation of the configuration $\boldsymbol{\mathfrak{C}}$ of $\boldsymbol{K}$ generated by the set $\boldsymbol{\mathfrak{B}}$.

Indeed, we take for $\rho(v)$ the restriction of the valuation v to the subfield $\boldsymbol{K}$ and we conclude using well-known results of the theory of valuations [8; 10].

Let us denote by $\boldsymbol{\mathfrak{T}}(\boldsymbol{\mathfrak{B}})$ the tree defined by the set of valuations $\boldsymbol{\mathfrak{B}}$ and by $\mathfrak{T}(\mathfrak{B})$ the tree defined by the set $\mathfrak{B}$ of their prolongations to K.

If $v \in \mathfrak{T}(\mathfrak{B})$, let $\mathfrak{F}(v) = \{v' \in \mathfrak{B} \mid v' \geqq v\}$; in this set we introduce the equivalence relation $v' \equiv v''$ (in $\mathfrak{F}(v)$) when v', v'' have the same restriction to $\boldsymbol{K}$.

We point out the following trivial facts: $\mathfrak{F}(v_0) = \mathfrak{B}$ (where v_0 is the trivial valuation); if $v_1, v_2 \in \mathfrak{T}(\mathfrak{B})$, $v_1 \neq v_2$, and have the same rank, then $\mathfrak{F}(v_1) \cap \mathfrak{F}(v_2) = \emptyset$; if $v_1, v_2 \in \mathfrak{T}(\mathfrak{B})$, $v_1 \leqq v_2$ then $\mathfrak{F}(v_1) \supseteq \mathfrak{F}(v_2)$.

For each valuation $\boldsymbol{v} \in \boldsymbol{\mathfrak{B}}$ let us denote by $\mathfrak{E}(\boldsymbol{v}) = \{v \in \mathfrak{B} \mid \rho(v) = \boldsymbol{v}\}$; then $\mathfrak{E}(\boldsymbol{v})$ is an equivalence class in $\mathfrak{F}(v_0) = \mathfrak{B}$.

Let $\mathfrak{W}$ be the set of all $w \in \mathfrak{T}(\mathfrak{B})$ having rank 1. In $\mathfrak{W}$ we define the equivalence relation: $w_1 \equiv w_2$ (in $\mathfrak{W}$) whenever their restrictions to $\boldsymbol{K}$ coincide. If $\boldsymbol{w} \in \boldsymbol{\mathfrak{T}}(\boldsymbol{\mathfrak{B}})$, and it has rank 1, let $\mathfrak{E}_1(\boldsymbol{w}) = \{w \in \mathfrak{W} \mid$ the restriction of w to $\boldsymbol{K}$ is $\boldsymbol{w}\}$; then $\mathfrak{E}_1(\boldsymbol{w})$ is an equivalence class in $\mathfrak{W}$.

Now, if $\boldsymbol{v} \in \boldsymbol{\mathfrak{B}}$, if $\boldsymbol{w}$ is the unique valuation of $\boldsymbol{K}$, of rank 1, $\boldsymbol{v} \geqq \boldsymbol{w}$, we have: the sets $\mathfrak{E}(\boldsymbol{v}) \cap \mathfrak{F}(w)$, where $w \in \mathfrak{E}_1(\boldsymbol{w})$, are pairwise disjoint,

$$\mathfrak{E}(\boldsymbol{v}) = \bigcup_{w_1 \in \mathfrak{E}_1(\boldsymbol{w})} (\mathfrak{E}(\boldsymbol{v}) \cap \mathfrak{F}(w)) \tag{2}$$

and $\mathfrak{E}(\boldsymbol{v}) \cap \mathfrak{F}(w) \neq \emptyset$ if and only if $w \in \mathfrak{E}_1(\boldsymbol{w})$.

We give now a proof of these last assertions. If $w, w' \in \mathfrak{E}_1(\boldsymbol{w})$, $w \neq w'$, then $\mathfrak{F}(w) \cap \mathfrak{F}(w') = \emptyset$, hence the sets $\mathfrak{E}(\boldsymbol{v}) \cap \mathfrak{F}(w)$ are pairwise disjoint; if $v \in \mathfrak{B}$ has restriction to $\boldsymbol{K}$ equal to $\boldsymbol{v}$, then the unique valuation of rank 1 $w, v \geqq w$, has restriction $\boldsymbol{w}$, so $w \in \mathfrak{E}_1(\boldsymbol{w})$ and $v \in \mathfrak{E}(\boldsymbol{v}) \cap \mathfrak{F}(w)$; finally, let $w \in \mathfrak{E}_1(\boldsymbol{w})$, then (cf. [10]), there must exist a valuation v of K whose restriction to $\boldsymbol{K}$ is $\boldsymbol{v}$ and such that $w \leqq v$, which shows that $\mathfrak{E}(\boldsymbol{v}) \cap \mathfrak{F}(w) \neq \emptyset$; conversely, if $v \in \mathfrak{E}(\boldsymbol{v}) \cap \mathfrak{F}(w)$, then the restriction of w is coarser than $\boldsymbol{v}$, hence coincides with $\boldsymbol{w}$.

Let us suppose, furthermore, that:

(i) each valuation $\mathfrak{v} \in \mathfrak{B}$ is discrete of rank r;

(ii) for each valuation v' *strictly* coarser than some of the valuations $v \in \mathfrak{V}$, the residue class field K/v' is a *separable* extension of $\boldsymbol{K}/\mathfrak{v}'$ (where $\mathfrak{v}'$ is the restriction of v' to $\boldsymbol{K}$); in particular, K is also a separable extension of $\boldsymbol{K}$.

If $v \in \mathfrak{T}(\mathfrak{V})$ and $\mathfrak{v}$ is its restriction to $\boldsymbol{K}$, we let $e(v) = (v(K):\mathfrak{v}(\boldsymbol{K})), f(v) = [K/v : \boldsymbol{K}/\mathfrak{v}]$; in particular, if v is the trivial valuation v_0, then $e(v_0) = 1$, $f(v_0) = n = [K:\boldsymbol{K}]$.

Then, with above hypothesis and notations, we have, for each $v' \in \mathfrak{T}(\mathfrak{V})$ and for each equivalence class $\mathfrak{E}$ in $\mathfrak{F}(v')$:

$$f(v') = \sum_{v \in \mathfrak{E}} \frac{e(v)}{e(v')} \cdot f(v) \tag{3}$$

(cf. [10]). In particular, when we consider the trivial valuation, we obtain for each valuation $\mathfrak{v} \in \mathfrak{B}$:

$$n = [K:\boldsymbol{K}] = \sum_{v \in \mathfrak{E}(\mathfrak{v})} e(v) \cdot f(v). \tag{4}$$

All these considerations suggest us to introduce some definitions and notations.

Let $\boldsymbol{\mathfrak{C}} = \{\boldsymbol{\mathfrak{T}}, \boldsymbol{\Gamma}, \boldsymbol{\mathfrak{K}}\}$ be a configuration, let $\mathfrak{C} = \{\mathfrak{T}, \Gamma, \mathfrak{K}\}$ be a prolongation of the configuration $\boldsymbol{\mathfrak{C}}, \rho$ the covering mapping.

For each $\gamma \in \mathfrak{T}$ let $\mathfrak{F}(\gamma) = \{\gamma' \in \mathfrak{T} \mid \gamma' \geqq \gamma, \gamma'$ maximal in $\mathfrak{T}\}$; in this set we introduce the equivalence relation $\gamma' \equiv \gamma''$ (in $\mathfrak{F}(\gamma)$) whenever $\rho(\gamma') = \rho(\gamma'')$.

(26) We have: $\mathfrak{F}(\delta) = \mathfrak{M}$ (set of all maximal elements of $\mathfrak{T}$); if $\gamma_1, \gamma_2 \in \mathfrak{T}$, $\gamma_1 \neq \gamma_2$ and γ_1, γ_2 have the same length, then $\mathfrak{F}(\gamma_1) \cap \mathfrak{F}(\gamma_2) = \emptyset$; if $\gamma_1, \gamma_2 \in \mathfrak{T}$, $\gamma_1 \leqq \gamma_2$ then $\mathfrak{F}(\gamma_1) \supseteq \mathfrak{F}(\gamma_2)$.

The proofs are straightforward.

For each $\boldsymbol{\gamma} \in \boldsymbol{\mathfrak{M}}$ (set of maximal elements of $\boldsymbol{\mathfrak{T}}$), let $\mathfrak{E}(\boldsymbol{\gamma}) = \{\gamma \in \mathfrak{T} \mid \rho(\gamma) = \boldsymbol{\gamma}\}$; hence each $\gamma \in \mathfrak{E}(\boldsymbol{\gamma})$ has length r and $\mathfrak{E}(\boldsymbol{\gamma})$ is an equivalence class in $\mathfrak{M}$.

Let $\mathfrak{B}$ be the set of all $\beta \in \mathfrak{T}$ with length 1. In $\mathfrak{B}$ we define the equivalence relation: $\beta_1 \equiv \beta_2$ (in $\mathfrak{B}$) whenever $\rho(\beta_1) = \rho(\beta_2)$. If $\boldsymbol{\beta} \in \boldsymbol{\mathfrak{T}}$, with length 1, let $\mathfrak{E}_1(\boldsymbol{\beta}) = \{\beta \in \mathfrak{B} \mid \rho(\beta) = \boldsymbol{\beta}\}$; then $\mathfrak{E}_1(\boldsymbol{\beta})$ is not empty and it is an equivalence class in $\mathfrak{B}$.

(27) Now, if $\boldsymbol{\gamma} \in \boldsymbol{\mathfrak{M}} \subseteq \boldsymbol{\mathfrak{T}}$, if $\boldsymbol{\beta}$ is the unique element of $\boldsymbol{\mathfrak{B}} \subseteq \boldsymbol{\mathfrak{T}}$ (of length 1), $\boldsymbol{\beta} \leqq \boldsymbol{\gamma}$, we have: the sets $\mathfrak{E}(\boldsymbol{\gamma}) \cap \mathfrak{F}(\beta)$, with $\rho(\beta) = \boldsymbol{\beta}$, are pairwise disjoint,

$$\mathfrak{E}(\boldsymbol{\gamma}) = \bigcup_{\beta \in \mathfrak{E}_1(\boldsymbol{\beta})} (\mathfrak{E}(\boldsymbol{\gamma}) \cap \mathfrak{F}(\beta))$$

and $\mathfrak{E}(\boldsymbol{\gamma}) \cap \mathfrak{F}(\beta) \neq \emptyset$ if and only if $\beta \in \mathfrak{E}_1(\boldsymbol{\beta})$.

Indeed, if β, $\beta' \in \mathfrak{E}_1(\boldsymbol{\beta})$, $\beta \neq \beta'$ then $\mathfrak{F}(\beta) \cap \mathfrak{F}(\beta') = \emptyset$, hence the sets $\mathfrak{E}(\boldsymbol{\gamma}) \cap \mathfrak{F}(\beta)$ are pairwise disjoint; if $\gamma \in \mathfrak{M}$, $\rho(\gamma) = \boldsymbol{\gamma}$ then for the unique element

$\beta \in \mathfrak{B}$ such that $\beta \leqq \gamma$, we have $\rho(\beta) \leqq \rho(\gamma) = \boldsymbol{\gamma}$, hence $\rho(\beta) = \boldsymbol{\beta}$ and $\beta \in \mathfrak{E}_1(\boldsymbol{\beta})$, $\gamma \in \mathfrak{E}(\boldsymbol{\gamma}) \cap \mathfrak{F}(\beta)$. Finally, let $\beta \in \mathfrak{E}_1(\boldsymbol{\beta})$, let $\gamma \in \mathfrak{E}(\boldsymbol{\gamma})$, hence $\rho(\beta) = \boldsymbol{\beta} \leqq \boldsymbol{\gamma} = \rho(\gamma)$, hence there exists (by property 17) $\gamma_1 \in \mathfrak{T}$ such that $\beta \leqq \gamma_1$ and $\rho(\gamma_1) = \rho(\gamma) = \boldsymbol{\gamma}$, that is $\gamma_1 \in \mathfrak{E}(\boldsymbol{\gamma}) \cap \mathfrak{F}(\beta)$; conversely, if $\gamma \in \mathfrak{E}(\boldsymbol{\gamma})$, $\gamma \geqq \beta$, then $\boldsymbol{\gamma} = \rho(\gamma) \geqq \rho(\beta)$, so that $\rho(\beta) = \boldsymbol{\beta}$ and $\beta \in \mathfrak{E}_1(\boldsymbol{\beta})$.

Definition. Let $\boldsymbol{\mathfrak{C}} = \{\boldsymbol{\mathfrak{T}}, \boldsymbol{\Gamma}, \boldsymbol{\mathfrak{K}}\}$, $\mathfrak{C} = \{\mathfrak{T}, \Gamma, \mathfrak{K}\}$, be configurations such that $\mathfrak{C}$ is a prolongation of $\boldsymbol{\mathfrak{C}}$. We say that $\mathfrak{C}$ is an *admissibble prolongation of* $\boldsymbol{\mathfrak{C}}$ whenever:

(28) for every $\gamma \in \mathfrak{T}$ the index $(\Gamma(\gamma) : \boldsymbol{\Gamma}(\rho(\gamma)))$ is finite, denoted by $e(\gamma)$;

(29) for every $\gamma \in \mathfrak{T}$, $\gamma \neq \delta$, the field $\mathfrak{K}(\gamma)$ is a separable extension of $\boldsymbol{\mathfrak{K}}(\rho(\gamma))$ of finite degree, denoted by $f(\gamma)$;

(30) for every $\gamma' \in \mathfrak{T}$, $\gamma' \neq \delta$ and for each equivalence class $\mathfrak{E}$ of $\mathfrak{F}(\gamma')$:

$$f(\gamma') = \sum_{\gamma \in \mathfrak{E}} \frac{e(\gamma)}{e(\gamma')} \cdot f(\gamma) \;; \tag{5}$$

(31) for any two equivalence classes $\mathfrak{E}, \mathfrak{E}'$ in $\mathfrak{F}(\delta) = \mathfrak{M}$ we have:

$$\sum_{\gamma \in \mathfrak{E}} e(\gamma) \cdot f(\gamma) = \sum_{\gamma' \in \mathfrak{E}'} e(\gamma') \cdot f(\gamma'). \tag{6}$$

2. **The theorem.** We want to now prove the following converse existence theorem:

Theorem. *Let $\boldsymbol{K}$ be a field, $\boldsymbol{\mathfrak{B}}$ a finite set of discrete valuations of rank $r \geqq 1$ of the field $\boldsymbol{K}$, let $\boldsymbol{\mathfrak{C}} = \{\boldsymbol{\mathfrak{T}}, \boldsymbol{v} \to \boldsymbol{v}(\boldsymbol{K}), \boldsymbol{v} \to \boldsymbol{K}/\boldsymbol{v}\}$ be the configuration of $\boldsymbol{K}$ generated by the given set of valuations. We suppose:*

(32) *for each $\boldsymbol{u} \in \boldsymbol{\mathfrak{T}}$ having rank strictly smaller than r, there exists a discrete valuation $\boldsymbol{v}'$ of rank r such that $\boldsymbol{v}' \wedge \boldsymbol{v} = \boldsymbol{u}$ for every $\boldsymbol{v} \in \boldsymbol{\mathfrak{B}}$, $\boldsymbol{v} > \boldsymbol{u}$.*

Let $\mathfrak{C}$ be a configuration which is an admissible prolongation of $\boldsymbol{\mathfrak{C}}$, and denote by $\rho : \mathfrak{T} \to \boldsymbol{\mathfrak{T}}$ the covering mapping of $\mathfrak{C}$ over $\boldsymbol{\mathfrak{C}}$.

Then, there exists a finite extension $K|\boldsymbol{K}$ such that:

(33) *$K|\boldsymbol{K}$ is separable;*

(34) *there exists an isomorphism I of the configuration $\mathfrak{C}$ onto the configuration of K generated by the set $\mathfrak{B}$ of valuations which extend the valuations of $\boldsymbol{\mathfrak{B}}$ to K, in such a way that $\rho(\gamma)$ is equal to the restriction to $\boldsymbol{K}$ of the valuation $I(\gamma)$ of K;*

(35) *for each valuation $\boldsymbol{v} \in \boldsymbol{\mathfrak{B}}$:*

$$[K : \boldsymbol{K}] = \sum e(v) \cdot f(v)$$

(sum over the set $I\mathfrak{E}(\boldsymbol{v})$ of valuations v of K which extend $\boldsymbol{v}$).

Proof. The theorem will be proved by induction on the rank r.

If $r = 1$ the theorem reduces to Krull's existence theorem [5], the hypothesis 32 being exactly Krull's hypothesis of the existence of a further discrete valuation of rank 1 in $\boldsymbol{K}$.

We now suppose that $r > 1$ and that the theorem is true for fields with the above properties and given valuations of rank at most $r-1$.

Let $\mathfrak{B}$ be the set of elements of length 1 in $\mathfrak{T}$.

For each $\beta \in \mathfrak{B}$ we shall define a configuration $\mathfrak{C}_\beta$.

Let $\mathfrak{T}_\beta = \{\gamma \in \mathfrak{T} \mid \beta \leqq \gamma\}$; $\mathfrak{T}_\beta$ is again a tree, with first element β and length $r-1 > 0$. We remark that $l_{\mathfrak{T}_\beta}(\gamma) = l_{\mathfrak{T}}(\gamma) - 1$ for every $\gamma \in \mathfrak{T}_\beta$.

Let Γ_β be a mapping that associates to each $\gamma \in \mathfrak{T}_\beta$ the following totally ordered abelian additive group: $\Gamma_\beta(\gamma) = \ker \theta_{\gamma,\beta}$.

Then Γ_β is a value function for the tree $\mathfrak{T}_\beta$. Indeed, $\Gamma_\beta(\gamma)$ is a group of rank $r-1$ if and only if γ is a maximal element of $\mathfrak{T}_\beta$, because $\Gamma(\beta)$ has rank 1, hence $\ker \theta_{\gamma,\beta}$ has rank $r-1$ whenever $\Gamma(\gamma)$ has rank r. If $\gamma' \leqq \gamma$ we define $(\theta_\beta)_{\gamma,\gamma'}$: $\Gamma_\beta(\gamma) \to \Gamma_\beta(\gamma')$ as the restriction of $\theta_{\gamma,\gamma'}$ to $\Gamma_\beta(\gamma)$; for that purpose, we must notice that from $\beta \leqq \gamma' \leqq \gamma$ it follows that $\theta_{\gamma,\beta} = \theta_{\gamma',\beta} \circ \theta_{\gamma,\gamma'}$, hence $\Gamma_\beta(\gamma) = \ker \theta_{\gamma,\beta} = \theta_{\gamma,\gamma'}^{-1}$ $(\ker\ \theta_{\gamma',\beta}) = \theta_{\gamma,\gamma'}^{-1}[\Gamma_\beta(\gamma')]$; on the other hand, $(\theta_\beta)_{\gamma,\gamma'}$ is onto $\Gamma_\beta(\gamma') = \ker\ \theta_{\gamma',\beta}$ as is shown by the commutative exact diagram below:

(7)
$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & 0 & \longrightarrow & \Gamma(\beta) & \xrightarrow{i_1} & \Gamma(\beta) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle \theta_{\gamma,\beta}} & & \uparrow{\scriptstyle \theta_{\gamma',\beta}} & & \\
0 & \longrightarrow & \ker \theta_{\gamma,\gamma'} & \xrightarrow{i_5} & \Gamma(\gamma) & \xrightarrow{\theta_{\gamma,\beta'}} & \Gamma(\gamma') & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle i_4} & & \uparrow{\scriptstyle i_3} & & \uparrow{\scriptstyle i_2} & & \\
0 & \longrightarrow & \ker (\theta_\beta)_{\gamma,\gamma'} & \xrightarrow{i_6} & \Gamma_\beta(\gamma) & \xrightarrow{(\theta_\beta)_{\gamma,\gamma'}} & \Gamma_\beta(\gamma') & \dashrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Indeed, let $a \in \Gamma_\beta(\gamma')$, then there exists $b \in \Gamma(\gamma)$ such that $\theta_{\gamma,\gamma'}(b) = i_2(a)$. But $0 = \theta_{\gamma',\beta} \circ i_2(a) = \theta_{\gamma',\beta} \circ \theta_{\gamma,\gamma'}(b) = i_1 \circ \theta_{\gamma,\beta}(b)$, hence $\theta_{\gamma,\beta}(b) = 0$, so we have $b = i_3(c)$ for some $c \in \Gamma_\beta(\gamma)$ and $i_2 \circ (\theta_\beta)_{\gamma,\gamma'}(c) = \theta_{\gamma,\gamma'} \circ i_3(c) = i_2(a)$ from which we conclude that $(\theta_\beta)_{\gamma,\gamma'}(c) = a$.

Moreover, $(\theta_\beta)_{\gamma,\gamma'}$ is an isomorphism if and only if $\gamma = \gamma'$; it is sufficient to show that the mapping i_4 is onto $\ker \theta_{\gamma,\gamma'}$. Let $a \in \ker \theta_{\gamma,\gamma'}$ and $i_5(a) = b$, then $\theta_{\gamma,\gamma'}(b) = 0$, hence $0 = \theta_{\gamma',\beta} \circ \theta_{\gamma,\gamma'}(b) = i_1 \circ \theta_{\gamma,\beta}(b)$, thus $\theta_{\gamma,\beta}(b) = 0$ and there exists $c \in \Gamma_\beta(\gamma)$ such that $i_3(c) = b$; it follows that $0 = \theta_{\gamma,\gamma'} \circ i_3(c) = i_2 \circ (\theta_{\beta\gamma,\gamma'}(c)$

hence $(\theta_\beta)_{\gamma,\gamma'}(c) = 0$ and there exists $d \in \ker(\theta_\beta)_{\gamma,\gamma'}$ such that $i_6(d) = c$; finally, $i_5 \circ i_4(d) = i_3 \circ i_6(d) = b = i_5(a)$, so $i_4(d) = a$.

Finally, if $\gamma'' \leqq \gamma' \leqq \gamma$ then $(\theta_\beta)_{\gamma,\gamma''} = (\theta_\beta)_{\gamma',\gamma''} \circ (\theta_\beta)_{\gamma,\gamma'}$ because those mappings are defined as restrictions of corresponding mappings $\theta_{\gamma,\gamma'}$, $\theta_{\gamma,\gamma''}$, $\theta_{\gamma',\gamma''}$.

Let now $\mathfrak{K}_\beta$ be the mapping that associates to each $\gamma \in \mathfrak{T}_\beta$, $\gamma \neq \beta$, the field $\mathfrak{K}_\beta(\gamma) = \mathfrak{K}(\gamma)$; we put also $(\pi_\beta)_\gamma/\gamma' = \pi_\beta/\gamma'$ whenever $\beta < \gamma' \leqq \gamma$.

Then, this mapping is a field function for the tree $\mathfrak{T}_\beta$, the conditions being trivially satisfied.

This being so, we now have, for every $\beta \in \mathfrak{B}$, a configuration $\mathfrak{C}_\beta = \{\mathfrak{T}_\beta, \Gamma_\beta, \mathfrak{K}_\beta\}$ whose tree has length $r - 1$.

Let $\mathfrak{W}$ be the set of valuations $w \in \mathfrak{T}$ having rank 1; hence, by 16, $\mathfrak{W} = \rho(\mathfrak{B})$. For each $w \in \mathfrak{W}$, let $\mathfrak{B}(w) = \{v \in \mathfrak{B} \mid v \geqq w\}$; then the sets $\mathfrak{B}(w)$ are pairwise disjoint and their union is $\mathfrak{B}$. For each $w \in \mathfrak{W}$ we denote also by $\mathfrak{E}_1(w) = \{\beta \in \mathfrak{B} \mid \rho(\beta) = w\}$, hence these sets are nonempty, pairwise disjoint and their union is $\mathfrak{B}$. Similarly, for each $v \in \mathfrak{B}$ let $\mathfrak{E}(v) = \{\gamma \in \mathfrak{T} \mid \rho(\gamma) = v\}$, let $w \in \mathfrak{W}$ be the unique valuation of rank 1 such that $w \leqq v$, then: $\beta \in \mathfrak{E}_1(w)$ if and only if $\mathfrak{E}(v) \cap \mathfrak{F}(\beta) \neq \emptyset$, and

$$\mathfrak{E}(v) = \bigcup_{\beta \in \mathfrak{E}_1(w)} (\mathfrak{E}(v) \cap \mathfrak{F}(\beta))$$

these sets being pairwise disjoint, by 27.

We remark also that the maximal ideal Q of the valuation ring of $w \in \mathfrak{W}$ is a prime ideal of the ring of every valuation $v \in \mathfrak{T}$, $v \geqq w$.

For each $\beta \in \mathfrak{B}$ let $\bar{K}_\beta = K/w$ be the residue class field of K with respect to $\rho(\beta) = w$ and for every $v \in \mathfrak{T}$, $v \geqq w$, let $\bar{v} = v/Q$; the fields $\bar{K}_\beta$, $\bar{K}_{\beta'}$ coincide when $\rho(\beta) = \rho(\beta')$.

Then $\bar{\mathfrak{T}}_\beta = \{\bar{v} = v/Q \mid v \in \mathfrak{T},\ v \geqq w = \rho(\beta)\}$ coincides with the tree of $\bar{K}_\beta$ generated by the set $\bar{\mathfrak{B}}_\beta = \{\bar{v} \mid v \in \mathfrak{B},\ v \geqq w = \rho(\beta)\}$ (cf. [6]).

We want to show now that the configuration $\mathfrak{C}_\beta$ is an admissible prolongation of the configuration $\bar{\mathfrak{C}}_\beta$ of $\bar{K}_\beta$, $\bar{\mathfrak{C}}_\beta = \{\bar{\mathfrak{T}}_\beta,\ \bar{v} \to \bar{v}(\bar{K}_\beta),\ \bar{v} \to \bar{K}_\beta/\bar{v}\}$, generated by the set $\bar{\mathfrak{B}}_\beta$ of valuations of $\bar{K}_\beta$.

Let $\rho_\beta : \mathfrak{T}_\beta \to \bar{\mathfrak{T}}_\beta$ be defined by $\rho_\beta(\gamma) = \rho(\gamma)/Q$. We have indeed $\rho_\beta(\gamma) \in \bar{\mathfrak{T}}_\beta$ because $\rho(\gamma) \in \mathfrak{T}$, $\gamma \geqq \beta$, implies $\rho(\gamma) \geqq \rho(\beta) = w$ hence $\rho(\gamma)/Q \in \bar{\mathfrak{T}}_\beta$.

(15β) The mapping ρ_β is clearly order-preserving and onto $\bar{\mathfrak{T}}_\beta$: if $\bar{v} \in \bar{\mathfrak{T}}_\beta$ then $\bar{v} = v/Q$, with $v \in \mathfrak{T} = \rho(\mathfrak{T})$, $v \geqq w = \rho(\beta)$; if $\gamma \in \mathfrak{T}$ is such that $\rho(\gamma) = v \geqq \rho(\beta)$, there exists $\gamma' \geqq \beta$ such that $\rho(\gamma') = \rho(\gamma) = v$ hence $\gamma' \in \mathfrak{T}_\beta$ and $\rho_\beta(\gamma') = \bar{v}$.

(16β) If $\gamma \in \mathfrak{T}_\beta$ then, by 16, $l_{\mathfrak{T}}(\gamma) = l_{\mathfrak{T}}(\rho(\gamma))$; as $l_{\mathfrak{T}\beta}(\gamma) = l_{\mathfrak{T}}(\gamma) - 1$ and $\rho_\beta(\gamma) = \rho(\gamma)/Q$ then $l_{\bar{\mathfrak{T}}\beta}(\rho_\beta(\gamma)) = l_{\mathfrak{T}}(\rho(\gamma)) - 1 = l_{\mathfrak{T}\beta}(\gamma)$.

(17β) Let $\gamma, \gamma' \in \mathfrak{T}_\beta$ and $\rho_\beta(\gamma) < \rho_\beta(\gamma')$; then $\rho(\gamma) < \rho(\gamma')$, hence (by 17) there exists $\gamma_1' \in \mathfrak{T}$, such that $\gamma < \gamma_1'$ and $\rho(\gamma_1') = \rho(\gamma')$, hence $\rho_\beta(\gamma_1') = \rho_\beta(\gamma')$; from $\beta \leqq \gamma < \gamma_1'$ it follows that $\gamma_1' \in \mathfrak{T}_\beta$.

Thus, we have established that $\mathfrak{T}_\beta$ is a prolongation of $\bar{\mathfrak{T}}_\beta$. We show now:

(22β) For every $\gamma \in \mathfrak{T}_\beta$ the value group of the valuation $\bar{\boldsymbol{v}} = \rho(\gamma)/\boldsymbol{Q}$ of $\bar{\boldsymbol{K}}_\beta$ is an ordered subgroup of the group $\Gamma_\beta(\gamma)$.

Indeed, $\bar{\boldsymbol{v}}(\bar{\boldsymbol{K}}_\beta) = \Delta$, isolated subgroup of $\boldsymbol{v}(\boldsymbol{K})$ (where $\boldsymbol{v} = \rho(\gamma) \geqq \boldsymbol{w}$) which corresponds to the prime ideal $\boldsymbol{Q}$ of the ring of the valuation $\boldsymbol{v}$. On the other hand, by definition $\Gamma_\beta(\gamma) = \ker \theta_{\gamma,\beta}$. By hypothesis, $\boldsymbol{v}(\boldsymbol{K})$ is an ordered subgroup of $\Gamma(\gamma)$, $\boldsymbol{w}(\boldsymbol{K})$ is an ordered subgroup of $\Gamma(\beta)$ and $\boldsymbol{\theta}_{\rho(\gamma),\rho(\beta)} = \boldsymbol{\theta}_{\boldsymbol{v},\boldsymbol{w}} : \boldsymbol{v}(\boldsymbol{K}) \to \boldsymbol{w}(\boldsymbol{K})$ is the restriction of $\theta_{\gamma,\beta}$ to $\boldsymbol{v}(\boldsymbol{K})$. Thus, we have the following commutative exact diagram:

$$
\text{(8)} \qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma_\beta(\gamma) & \xrightarrow{i_1} & \Gamma(\gamma) & \xrightarrow{\theta_{\gamma,\beta}} & \Gamma_{(\beta)} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \lambda} & & \uparrow{\scriptstyle i_3} & & \uparrow{\scriptstyle i_4} & & \\
0 & \longrightarrow & \bar{\boldsymbol{v}}(\bar{\boldsymbol{K}}_\beta) & \xrightarrow{i_2} & \boldsymbol{v}(\boldsymbol{K}) & \xrightarrow{\boldsymbol{\theta}_{\boldsymbol{v},\boldsymbol{w}}} & \boldsymbol{w}(\boldsymbol{K}) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & &
\end{array}
$$

Indeed, let $a \in \bar{\boldsymbol{v}}(\bar{\boldsymbol{K}}_\beta) = \Delta$, $b = i_2(a)$, then $\boldsymbol{\theta}_{\boldsymbol{v},\boldsymbol{w}}(b) = 0$ hence $0 = i_4 \circ \boldsymbol{\theta}_{\boldsymbol{v},\boldsymbol{w}}(b) = \theta_{\gamma,\beta} \circ i_3(b)$ hence there exists $c \in \Gamma_\beta(\gamma)$ such that $i_1(c) = i_3(b) = i_3 \circ i_2(a)$; we define $\lambda(a) = c$; thus $i_1 \circ \lambda(a) = i_3 \circ i_2(a)$, the diagram is then commutative; λ is clearly an order-homomorphism and finally, if $\lambda(a) = 0$ then $0 = i_1 \circ \lambda(a) = i_3 \circ i_2(a)$ hence $a = 0$.

(23β) If $\gamma' \leqq \gamma$ in $\mathfrak{T}_\beta$ then $(\theta_\beta)_{\rho_\beta(\gamma),\rho_\beta(\gamma')}$ is the restriction of $(\theta_\beta)_{\gamma,\gamma'}$ to the value group of $\bar{\boldsymbol{v}} = \rho(\gamma)/\boldsymbol{Q}$.

Let $\boldsymbol{v} = \rho(\gamma)$, $\boldsymbol{v}' = \rho(\gamma')$, $\bar{\boldsymbol{v}}' = \boldsymbol{v}'/\boldsymbol{Q}$. By hypothesis, $\boldsymbol{v}(\boldsymbol{K})$ is an ordered subgroup of $\Gamma(\gamma)$, $\boldsymbol{v}'(\boldsymbol{K})$ is an ordered subgroup of $\Gamma(\gamma')$, $\boldsymbol{\theta}_{\boldsymbol{v},\boldsymbol{v}'}$ is the restriction of $\theta_{\gamma,\gamma'}$ to $\boldsymbol{v}(\boldsymbol{K})$.

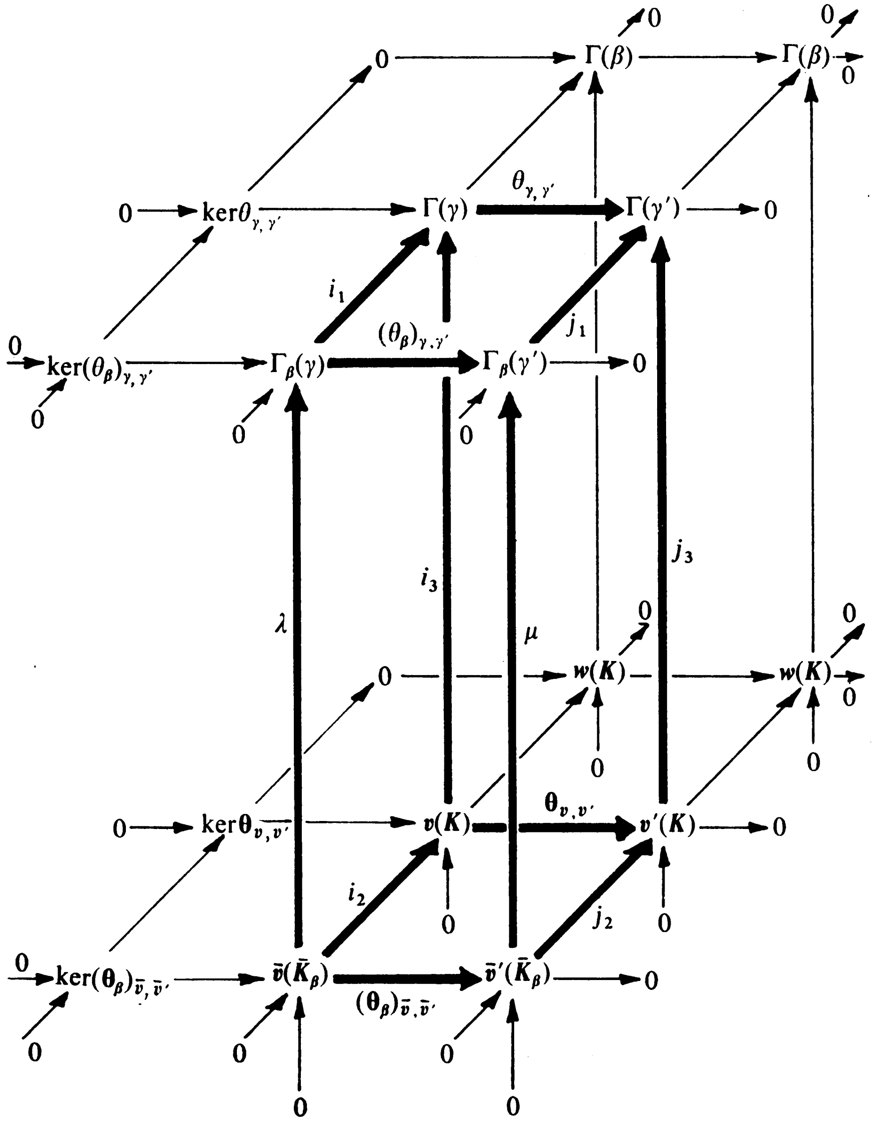
Hence, the following diagram on the next page is exact and commutative.

Indeed, we want to show that $\mu \circ (\boldsymbol{\theta}_\beta)_{\bar{\boldsymbol{v}},\bar{\boldsymbol{v}}'} = (\theta_\beta)_{\gamma,\gamma'} \circ \lambda$. Let $a \in \bar{\boldsymbol{v}}(\bar{\boldsymbol{K}}_\beta)$ then $j_1 \circ \mu \circ (\boldsymbol{\theta}_\beta)_{\bar{\boldsymbol{v}},\bar{\boldsymbol{v}}'}(a) = j_3 \circ j_2 \circ (\boldsymbol{\theta}_\beta)_{\bar{\boldsymbol{v}},\bar{\boldsymbol{v}}'}(a) = j_3 \circ \boldsymbol{\theta}_{\bar{\boldsymbol{v}},\bar{\boldsymbol{v}}'} \circ i_2(a) = \theta_{\gamma,\gamma'} \circ i_3 \circ i_2(a) = \theta_{\gamma,\gamma'} \circ i_1 \circ \lambda(a) = j_1 \circ (\theta_\beta)_{\gamma,\gamma'} \circ \lambda(a)$ hence $\mu \circ (\boldsymbol{\theta}_\beta)_{\bar{\boldsymbol{v}},\bar{\boldsymbol{v}}'}(a) = (\theta_\beta)_{\gamma,\gamma'} \circ \lambda(a)$.

(24β) Let $\gamma \in \mathfrak{T}_\beta$, $\gamma \neq \beta$; then $\mathfrak{K}_\beta(\gamma) = \mathfrak{K}(\gamma)$. On the other hand, the residue class field of $\boldsymbol{K}/\boldsymbol{w} = \bar{\boldsymbol{K}}_\beta$ by the valuation $\bar{\boldsymbol{v}} = \boldsymbol{v}/\boldsymbol{Q}$ is the same as the residue class field of $\boldsymbol{K}$ by the valuation $\boldsymbol{v} : \bar{\boldsymbol{K}}_\beta/\bar{\boldsymbol{v}} = \boldsymbol{K}/\boldsymbol{v}$. By hypothesis, $\mathfrak{K}(\gamma) = \mathfrak{K}_\beta(\gamma)$ is an extension of the field $\boldsymbol{K}/\boldsymbol{v} = \bar{\boldsymbol{K}}_\beta/\bar{\boldsymbol{v}}$.

(25β) Let $\beta < \gamma' \leqq \gamma$, $\rho(\gamma') = \boldsymbol{v}'$, $\rho(\gamma) = \boldsymbol{v}$, $\rho_\beta(\gamma') = \boldsymbol{v}'/\boldsymbol{Q} = \bar{\boldsymbol{v}}'$, $\rho_\beta(\gamma) = \boldsymbol{v}/\boldsymbol{Q} = \bar{\boldsymbol{v}}$. We show now that the place $\boldsymbol{\pi}_{\bar{\boldsymbol{v}}}/\bar{\boldsymbol{v}}'$ of $\bar{\boldsymbol{K}}_\beta/\bar{\boldsymbol{v}}' = \boldsymbol{K}/\boldsymbol{v}'$ (associated with the valuation $\bar{\boldsymbol{v}}/\bar{\boldsymbol{P}}$, where $\boldsymbol{v}' = \boldsymbol{v}_{\boldsymbol{P}}$ and $\bar{\boldsymbol{P}} = \boldsymbol{P}/\boldsymbol{Q}$) is the restriction of the place π_γ/γ' of $\mathfrak{K}_\beta(\gamma') = \mathfrak{K}(\gamma')$.

(9)



Indeed, we remark that $\bar{v}/\bar{P} = v/P$ and by hypothesis the restriction of the place π_γ/γ': $\mathfrak{K}(\gamma') \to \mathfrak{K}(\gamma)$ is the place associated to v/P; we conclude noting that $\mathfrak{K}_\beta(\gamma') = \mathfrak{K}(\gamma')$, $\mathfrak{K}_\beta(\gamma) = \mathfrak{K}(\gamma)$ and $(\pi_\beta)_\gamma/\gamma' = \pi_\gamma/\gamma'$.

We know now that $\mathfrak{C}_\beta$ is a prolongation of $\bar{\mathfrak{C}}_\beta$, and we proceed to prove that it is an admissible prolongation.

(28β) For every $\gamma \in \mathfrak{T}_\beta$ the index $(\Gamma_\beta(\gamma) : \bar{v}(\bar{K}_\beta))$ is finite (where $v = \bar{v}/Q$, $v = \rho(\gamma)$).

Indeed, from the exact diagram (8) we deduce:

$$(\Gamma(\gamma) : v(K)) = (\Gamma_\beta(\gamma) : \bar{v}(\bar{K}_\beta)) \cdot (\Gamma(\beta) : w(K)) \tag{10}$$

and the finiteness of $(\Gamma_\beta(\gamma) : \bar{v}(\bar{K}_\beta))$ comes from the hypothesis which implies that $(\Gamma(\gamma) : v(K))$ is finite.

(29β) For every $\gamma \in \mathfrak{T}_\beta$, $\gamma \neq \beta$, the extension $\mathfrak{K}_\beta(\gamma) \,|\, (\bar{K}_\beta/\bar{v})$ is of finite degree (where $v = \rho(\gamma)$, $\bar{v} = v/Q$) and separable.

It is an immediate consequence of the hypothesis, noticing that $\mathfrak{K}_\beta(\gamma) = \mathfrak{K}(\gamma)$ and $\bar{K}_\beta/\bar{v} = K/v$.

(30β) For each $\gamma \geqq \beta$, let $\mathfrak{F}_\beta(\gamma) = \{\gamma' \in \mathfrak{T}_\beta \,|\, \gamma \leqq \gamma',\ \gamma' \text{ maximal in } \mathfrak{T}_\beta\}$; let $\gamma' \equiv \gamma''$ in $\mathfrak{F}_\beta(\gamma)$ whenever $\rho_\beta(\gamma') = \rho_\beta(\gamma'')$. We want to show that for every $\gamma \in \mathfrak{T}_\beta$, $\gamma \neq \beta$, and for every equivalence class $\mathfrak{E}_\beta$ in $\mathfrak{F}_\beta(\gamma)$ we have

$$(11) \qquad [\mathfrak{K}_\beta(\gamma) : (\bar{K}_\beta/\bar{v})] = \sum_{\gamma' \in \mathfrak{E}_\beta} \frac{(\Gamma_\beta(\gamma') : \bar{v}'(\bar{K}_\beta))}{(\Gamma_\beta(\gamma) : \bar{v}(\bar{K}_\beta))} \cdot [\mathfrak{K}_\beta(\gamma') : (\bar{K}_\beta/(\bar{v}')].$$

This will follow from the hypothesis, if we observe successively that $\mathfrak{K}_\beta(\gamma) = \mathfrak{K}(\gamma)$, $\bar{K}_\beta/\bar{v} = K/v$, $\mathfrak{K}_\beta(\gamma') = \mathfrak{K}(\gamma)$, $\bar{K}_\beta/\bar{v}' = K/v'$, by using the relation (10) and finally by remarking that $\mathfrak{F}_\beta(\gamma) = \mathfrak{F}(\gamma)$ and also each equivalence class $\mathfrak{E}_\beta$ coincides with an equivalence class $\mathfrak{E}$, because $\rho(\gamma) = \rho(\gamma')$ if and only if $\rho_\beta(\gamma) = \rho_\beta(\gamma')$.

(31β) Let $\mathfrak{E}_\beta$, $\mathfrak{E}'_\beta$ be two equivalence classes in $\mathfrak{F}_\beta(\beta) = \mathfrak{F}(\beta)$; we have then:

$$(12) \quad \sum_{\gamma \in \mathfrak{E}_\beta} (\Gamma_\beta(\gamma) : \bar{v}(\bar{K}_\beta)) \cdot [\mathfrak{K}_\beta(\gamma) : \bar{K}_\beta/\bar{v}] = \sum_{\gamma' \in \mathfrak{E}_{\beta'}} (\Gamma_\beta(\gamma') : \bar{v}'(\bar{K}_\beta)) \cdot [\mathfrak{K}_\beta(\gamma') : \bar{K}_\beta/\bar{v}'].$$

Indeed, by hypothesis 30 and above considerations, we have

$$(13) \qquad [\mathfrak{K}(\beta) : K/w] = \sum_{\gamma \in \mathfrak{E}} \frac{(\Gamma(\gamma) : v(K))}{(\Gamma(\beta) : w(K))} \cdot [\mathfrak{K}(\gamma) : K/v]$$

where $\mathfrak{E}$ is any equivalence class of $\mathfrak{F}(\beta)$ (which coincides with an equivalence class of $\mathfrak{F}_\beta(\beta)$), hence by the relation (10) and above remarks we deduce that:

$$(14) \qquad [\mathfrak{K}(\beta) : K/w] = \sum_{\gamma \in \mathfrak{E}} (\Gamma_\beta(\gamma) : \bar{v}(\bar{K}_\beta)) \cdot [\mathfrak{K}_\beta(\gamma) : \bar{K}_\beta/\bar{v}].$$

The left-hand side being independent of $\mathfrak{E}$, and the equivalence classes in $\mathfrak{F}_\beta(\beta)$ being classes in $\mathfrak{F}(\beta) = \mathfrak{F}_\beta(\beta)$, we have indeed the desired relation (12).

(32β) We remark now that the tree $\bar{\mathfrak{T}}_\beta$ of length $r-1$ verifies the condition 32 in the hypothesis of the theorem.

Indeed, let $\bar{u} \in \bar{\mathfrak{T}}_\beta$ have a rank strictly smaller than $r-1$; we have $\bar{u} = u/Q$ where $u \in \mathfrak{T}$, $u \geqq w = \rho(\beta)$ and u has rank strictly smaller than r. As $\mathfrak{T}$ satisfies 32, there exists a discrete valuation v' of rank r such that $v' \wedge v = u$ for every $v \in \mathfrak{V}$, $v > u$; we conclude that $v' \geqq w$ and $\bar{v}' = v'/Q$ is a discrete valuation of rank $r-1$ of $\bar{K}_\beta$, such that $\bar{v}' \wedge \bar{v} = \bar{u}$ for every $\bar{v} = v/Q$, $\bar{v} \in \mathfrak{V}_\beta$, $\bar{v} \geqq \bar{u}$.

Hence, by the induction hypothesis, for each $\beta \in \mathfrak{B}$, there exists a finite extension $K_\beta \mid \bar{K}_\beta$ such that:

(33β) $K_\beta \mid \bar{K}_\beta$ is separable;

(34β) there exists an isomorphism I_β of the configuration $\mathfrak{C}_\beta$ onto the configuration of K_β generated by the set $\mathfrak{V}_\beta$ of valuations of K_β which extend the valuations of $\bar{K}_\beta$ belonging to the set $\bar{\mathfrak{V}}_\beta = \{\bar{v} = v/Q \mid v \in \mathfrak{V}, v \geqq w = \rho(\beta)\}$; moreover, $\rho_\beta(\gamma)$ is equal to the restriction to $\bar{K}_\beta$ of the valuation $I_\beta(\gamma)$ of K_β;

(35β) for each valuation $\bar{v} \in \bar{\mathfrak{V}}_\beta$ we have:

$$[K_\beta : \bar{K}_\beta] = \sum (\bar{v}(K_\beta) : \bar{v}(\bar{K}_\beta)) \cdot [K_\beta/\bar{v} : \bar{K}_\beta/\bar{v}] \tag{15}$$

(this sum being over the set $I_\beta \mathfrak{C}_\beta(\bar{v})$ of valuations $\bar{v}$ of K_β which extend $\bar{v}$).

We now turn to the consideration of the tree of length 1 $\mathfrak{T}^* = \{\delta\} \cup \mathfrak{B}$. We define a configuration $\mathfrak{C}^* = \{\mathfrak{T}^*, \Gamma^*, \mathfrak{K}^*\}$.

For this purpose, let Γ^* be the restriction of Γ to the tree $\mathfrak{T}^*$ and $\theta^*_{\beta,\delta}$ be the zero homomorphism of $\Gamma^*(\beta)$, while $\theta^*_{\beta,\beta}$, $\theta^*_{\delta,\delta}$ are the identity isomorphism of corresponding groups. Then, Γ^* is clearly a value function of $\mathfrak{T}^*$.

We define $\mathfrak{K}^*(\beta) = K_\beta$ for every $\beta \in \mathfrak{B}$ and we let π^*_β/β be the trivial place of K_β. Then $\mathfrak{K}^*$ is a field function of $\mathfrak{T}^*$ and this shows that $\mathfrak{C}^* = \{\mathfrak{T}^*, \Gamma^*, \mathfrak{K}^*\}$ is indeed a configuration.

We consider now the tree $\mathfrak{T}^*$ of K generated by the set $\mathfrak{W} = \{w \in \mathfrak{T} \mid w$ with rank 1$\}$ and we let $\mathfrak{C}^* = \{\mathfrak{T}^*, w \to w(K), w \to K/w\}$ be the configuration of K generated by the set $\mathfrak{W}$.

We proceed to prove that $\mathfrak{C}^*$ is a prologation of $\mathfrak{C}^*$.

We remark that ρ maps $\mathfrak{T}^*$ onto $\mathfrak{T}^*$ and satisfies obviously the properties which imply that $\mathfrak{T}^*$ is a prolongation of $\mathfrak{T}^*$.

As the mapping Γ^* is the restriction of Γ to $\mathfrak{T}^*$ and $\theta_{\beta,\delta}$ is also the zero homomorphism, then properties 22*, 23* of the definition of prolongation of a configuration are satisfied.

By construction, $K_\beta = \mathfrak{K}^*(\beta)$ is an extension of $\bar{K}_\beta = K/w$ (where $w = \rho(\beta)$) and property 25* is also trivially verified.

Finally, $\mathfrak{C}^*$ is an admissible prolongation of $\mathfrak{C}^*$. Property 28* is trivial, property 29* is satisfied by construction, as $K_\beta \mid \bar{K}_\beta$ is a separable extension of finite degree; and property 30* is also trivial.

We now prove property 31*: if $\mathfrak{E}^*$, $\mathfrak{E}^{*\prime}$ are equivalence classes in $\mathfrak{F}^*(\delta) = \mathfrak{B}$ we have:

$$\sum_{\beta \in \mathfrak{E}^*} (\Gamma(\beta): w(K)) \cdot [\mathfrak{K}^*(\beta):K/w] = \sum_{\beta' \in \mathfrak{E}^{*\prime}} (\Gamma(\beta') \cdot w'(K)) \cdot [\mathfrak{K}^*(\beta) : K/w']$$

where $\rho(\beta) = w$ for every $\beta \in \mathfrak{E}^*$, $\rho(\beta') = w'$ for every $\beta' \in \mathfrak{E}^{*\prime}$.

We remark that $\mathfrak{E}^* = \mathfrak{E}_1(w)$ and similarly $\mathfrak{E}^{*\prime} = \mathfrak{E}_1(w')$; as $\mathfrak{K}^*(\beta) = K_\beta$, $K/w = \bar{K}_\beta$, we have to compute $[K_\beta : \bar{K}_\beta]$.

We have already seen at the relation (15) that for every valuation $\bar{\mathfrak{v}} = \mathfrak{v}/Q$, $\mathfrak{v} \geqq \mathfrak{w}$, of $\bar{K}_\beta$:

$$[K_\beta : \bar{K}_\beta] = \sum (\bar{v}(K_\beta) : \bar{\mathfrak{v}}(\bar{K}_\beta)) \cdot [K_\beta/\bar{v} : \bar{K}_\beta/\bar{\mathfrak{v}}],$$

sum over the set $I_\beta \mathfrak{E}_\beta(\bar{\mathfrak{v}})$ of valuations $\bar{v}$ of K_β which extend $\bar{\mathfrak{v}}$.

By the isomorphism I_β of the configuration $\mathfrak{C}_\beta$ onto the configuration of K_β generated by the set $\mathfrak{V}_\beta$ of valuations of K_β which extend those in $\bar{\mathfrak{V}}_\beta$, we deduce that $\gamma \in \mathfrak{E}(\mathfrak{v}) \cap \mathfrak{F}(\beta)$ (that is $\rho(\gamma) = \mathfrak{v}$, $\beta \leqq \gamma$ in $\mathfrak{T}$) if and only if $I_\beta(\gamma) = \bar{v} \in I_\beta \mathfrak{E}_\beta(\bar{\mathfrak{v}})$ because $\bar{\mathfrak{v}}$ is equal to the restriction $\rho_\beta(\gamma)$ of $I_\beta(\gamma)$ if and only if $\rho(\gamma) = \mathfrak{v}$, $\beta \leqq \gamma$.

If $I_\beta(\gamma) = \bar{v} \in \mathfrak{V}_\beta$ we have $\bar{v}(K_\beta) = \Gamma_\beta(\gamma)$, by 13, hence

$$(\bar{v}(K_\beta) : \bar{\mathfrak{v}}(\bar{K}_\beta)) = (\Gamma_\beta(\gamma) : \bar{\mathfrak{v}}(\bar{K}_\beta)).$$

Similarly, if $I_\beta(\gamma) = \bar{v} \in \mathfrak{V}_\beta$ we have $\mathfrak{K}_\beta(\gamma) = K_\beta/\bar{v}$ (by (14)) hence $[K_\beta/\bar{v} : \bar{K}_\beta/\bar{\mathfrak{v}}] = [\mathfrak{K}_\beta(\gamma) : \bar{K}_\beta/\bar{\mathfrak{v}}]$.

This shows that

$$[K_\beta : \bar{K}_\beta] = \sum (\Gamma_\beta(\gamma) : \bar{\mathfrak{v}}(\bar{K}_\beta)) \cdot [\mathfrak{K}_\beta(\gamma) : \bar{K}_\beta/\bar{\mathfrak{v}}]$$
$$= \sum \frac{(\Gamma(\gamma) : \mathfrak{v}(\boldsymbol{K}))}{(\Gamma(\beta) : \mathfrak{w}(\boldsymbol{K}))} \cdot [\mathfrak{K}(\gamma) : \boldsymbol{K}/\mathfrak{v}]$$

(sum over the set $\mathfrak{E}(\mathfrak{v}) \cap \mathfrak{F}(\beta)$), because of relation (10) and $\mathfrak{K}_\beta(\gamma) = \mathfrak{K}(\gamma)$, $\bar{K}_\beta/\bar{\mathfrak{v}} = \boldsymbol{K}/\mathfrak{v}$. So,

$$\sum_{\beta \in \mathfrak{C}^*} (\Gamma(\beta) : \mathfrak{w}(\boldsymbol{K})) \cdot [\mathfrak{K}^*(\beta) : \boldsymbol{K}/\mathfrak{w}]$$
$$= \sum_{\beta \in \mathfrak{E}_1(\mathfrak{w})} \sum_{\gamma \in \mathfrak{E}(\mathfrak{v}) \cap \mathfrak{F}(\beta)} (\Gamma(\gamma) : \mathfrak{v}(\boldsymbol{K})) \cdot [\mathfrak{K}(\gamma) : \boldsymbol{K}/\mathfrak{v}]$$
$$= \sum_{\gamma \in \mathfrak{E}(\mathfrak{v})} (\Gamma(\gamma) : \mathfrak{v}/\boldsymbol{K})) \cdot [\mathfrak{K}(\gamma) : \boldsymbol{K}/\mathfrak{v}],$$

by 27. But, by hypothesis 31, this last sum is in fact independent of the equivalence class $\mathfrak{E}(\mathfrak{v})$, which shows that $\mathfrak{C}^*$ satisfies property 31*.

We remark now that $\mathfrak{T}^*$ satisfies the condition 32* in the hypothesis of the theorem. We have to prove that there exists a discrete valuation $\mathfrak{w}'$ of rank 1 in $\boldsymbol{K}$ such that $\mathfrak{w}' \notin \mathfrak{W}$. By hypothesis 32, given the trivial valuation of $\boldsymbol{K}$, there exists a discrete valuation $\mathfrak{v}'$ of rank r such that $\mathfrak{v}' \wedge \mathfrak{v}$ is the trivial valuation for every $\mathfrak{v} \in \mathfrak{V}$. We let $\mathfrak{w}'$ be the unique valuation of rank 1, coarser than $\mathfrak{v}'$; $\mathfrak{w}'$ is discrete and $\mathfrak{w}' \notin \mathfrak{W}$.

Hence, the theorem being true for the configuration $\mathfrak{C}^*$ of length 1, there exists a finite extension $K | \boldsymbol{K}$ such that:

(33*) $K | \boldsymbol{K}$ is separable;

(34*) there exists an isomorphism I^* of the configuration $\mathfrak{C}^*$ onto the configuration of K generated by the set $\mathfrak{W}$ of valuations of K which extend the valuations of $\mathfrak{W}$, in such a way that $\rho(\gamma)$ is equal to the restriction to $\boldsymbol{K}$ of the valuation $I^*(\gamma)$ of K;

(35*) for every valuation $\boldsymbol{w} \in \mathfrak{W}$:

$$[K:\boldsymbol{K}] = \sum (w(K):\boldsymbol{w}(\boldsymbol{K})) \cdot [K/w:\boldsymbol{K}/\boldsymbol{w}]$$

(this sum being over the set $I^*\mathfrak{E}^*(\boldsymbol{w})$ of valuations w of K which extend $\boldsymbol{w}$).

We now want to conclude the proof of the theorem by putting together the partial results already obtained.

For that purpose, we define an isomorphism I of the configuration $\mathfrak{C}$ onto the configuration of K generated by the set $\mathfrak{V}$ of valuations which extend the valuations of $\mathfrak{B}$.

We let $I(\delta)$ be the trivial valuation of K, $I(\beta) = I^*(\beta) = w$ for every $\beta \in \mathfrak{B}$ and finally, if $\gamma \in \mathfrak{T}$, $\gamma \geqq \beta$, we let $I(\gamma)$ be the unique valuation v of K, $v \geqq w = I^*(\beta)$ such that if $w = v_Q$ then $v/Q = I_\beta(\gamma)$.

The mapping $I:\mathfrak{T} \to \mathfrak{T}(\mathfrak{V})$ (tree generated by the set $\mathfrak{V}$ of valuations of K extending the valuations of $\mathfrak{B}$) so defined is one-to-one and preserves the order. It is also onto $\mathfrak{T}(\mathfrak{V})$, because if $v \in \mathfrak{T}(\mathfrak{V})$ there exists a unique valuation w of rank 1, $w \leqq v$, of K; let $\beta \in \mathfrak{B}$ be such that $I^*(\beta) = I(\beta) = w$, then $K/w = \mathfrak{K}^*(\beta) = K_\beta$; let $\bar{v}$ be the valuation of K/w corresponding uniquely to v and let $\gamma \in \mathfrak{C}_\beta$ be such that $I_\beta(\gamma) = \bar{v}$; then $v = I(\gamma)$.

It is also true that for every $\gamma \in \mathfrak{T}$, $\rho(\gamma)$ is the restriction to $\boldsymbol{K}$ of the valuation $I(\gamma)$ of K; this is trivial for δ and for each $\beta \in \mathfrak{B}$; if $\gamma \in \mathfrak{T}$, $\gamma \geqq \beta$, as $\rho_\beta(\gamma)$ is the restriction of $\bar{v} = v/Q = I_\beta(\gamma)$ to $\boldsymbol{K}/\rho(\beta) = \bar{\boldsymbol{K}}_\beta$, and v is the unique valuation of K corresponding to v, $\rho(\gamma)$ is the unique valuation of $\boldsymbol{K}$ corresponding to $\rho_\beta(\gamma)$, it follows that $\rho(\gamma)$ is the restriction of $I(\gamma)$ to $\boldsymbol{K}$.

We must finally prove that for every valuation $\boldsymbol{v} \in \mathfrak{B}$ we have:

$$[K:\boldsymbol{K}] = \sum_{v \in I\mathfrak{E}(\boldsymbol{v})} (v(K):\boldsymbol{v}(\boldsymbol{K})) \cdot [K/v:\boldsymbol{K}/\boldsymbol{v}].$$

To obtain this relation, we make use of the relations already obtained:

$$\text{(35*)} \qquad [K:\boldsymbol{K}] = \sum_{w \in I^*\mathfrak{E}^*(\boldsymbol{w})} (w(K):\boldsymbol{w}(\boldsymbol{K})) \cdot [K/w:\boldsymbol{K}/\boldsymbol{w}];$$

noticing that $\mathfrak{E}^*(\boldsymbol{w}) = \mathfrak{E}_1(\boldsymbol{w})$, $K/w = K_\beta$, $\boldsymbol{K}/\boldsymbol{w} = \bar{\boldsymbol{K}}_\beta$ (where $w = I(\beta)$) and that we have

$$\text{(35}\beta\text{)} \qquad \begin{aligned} [K_\beta:\bar{\boldsymbol{K}}_\beta] &= \sum_{\bar{v} \in I_\beta\mathfrak{E}_\beta(\bar{\boldsymbol{v}})} (\bar{v}(K_\beta):\bar{\boldsymbol{v}}(\bar{\boldsymbol{K}}_\beta)) \cdot [K_\beta/\bar{v}:\bar{\boldsymbol{K}}_\beta/\bar{\boldsymbol{v}}] \\ &= \sum_{v \in I\mathfrak{E}(\boldsymbol{v}) \cap \mathfrak{F}(w)} \frac{(v(K)):\boldsymbol{v}(\boldsymbol{K}))}{(w(K)):\boldsymbol{w}(\boldsymbol{K}))} \cdot [K/v:\boldsymbol{K}/\boldsymbol{v}] \end{aligned}$$

we conclude by 27 that

$$[K:\boldsymbol{K}] = \sum_{w \in I\mathfrak{E}_1(w)} \sum_{v \in I\,\mathfrak{E}(v) \cap \mathfrak{F}(w)} (v(K):\boldsymbol{v}(\boldsymbol{K})) \cdot [K/v:\boldsymbol{K}/\boldsymbol{v}]$$

$$= \sum_{v \in I\mathfrak{E}(v)} (v(K):\boldsymbol{v}(\boldsymbol{K})) \cdot [K/v:\boldsymbol{K}/\boldsymbol{v}].$$

This finishes the proof of the theorem.

3. Comments on the theorem.

(36) There is no loss of generality supposing that the given set of valuations (of finite rank) of $\boldsymbol{K}$ is such that every valuation has the same rank, provided we make a reasonable assumption on the existence of sufficiently many valutions of given rank in the field.

In fact, in any case, let $\boldsymbol{v} \in \mathfrak{B}$ be of maximal rank r and suppose that for every $\boldsymbol{v}_1 \in \mathfrak{B}$ having rank strictly smaller than r there exists a valuation $\boldsymbol{u}_1$ of $\boldsymbol{K}$, of rank r, such that $\boldsymbol{u}_1 > \boldsymbol{v}_1$. The set $\mathfrak{B}_1$ of valuations of rank r so obtained is such that $\mathfrak{T}(\mathfrak{B}) \subseteq \mathfrak{T}(\mathfrak{B}_1)$; moreover, we assume that a modified hypothesis 32 is satisfied for $\mathfrak{B}_1$. Then, the theorem is true for $\mathfrak{B}_1$, implying its modified validity for $\mathfrak{B}$.

(37) The significance of the theorem may be roughly expressed in the following way: apart from the already known properties and relations satisfied by the residue class fields, value groups, inertial degrees and ramification indices of the prolongations of a discrete valuation of finite rank to a finite separable extension, no other property or relation may be expected to hold in general; furthermore, given a tree generated by a finite set of valuations of $\boldsymbol{K}$, the tree generated by their prolongations may be given at will, provided the necessary conditions are satisfied.

In other words, no simplification in the general theory of prolongation of Krull valuations should be expected.

(38) About the hypothesis 32, we recall a result by F. K. Schmidt [13], for valuations of rank 1, which we have generalized for valuations of finite rank (cf. [7]):

Let $\boldsymbol{K}$ be a field, complete with respect to a discrete valuation $\boldsymbol{v}$ of finite rank r, if $\boldsymbol{v}'$ is a discrete valuation of $\boldsymbol{K}$ of rank r, then $\boldsymbol{v}' = \boldsymbol{v}$ (up to equivalence); similarly, if $\boldsymbol{K}$ is complete with respect to a valuation $\boldsymbol{w}'$ of rank r, then $\boldsymbol{w}' = \boldsymbol{v}$ (up to equivalence).

Hence, if hypothesis 32 is satisfied, then $\boldsymbol{K}$ is not complete with respect to any valuation $\boldsymbol{v} \in \mathfrak{B}$.

Thus, if $\boldsymbol{K}$ is complete with respect to a discrete valuation $\boldsymbol{v}$ of rank r, the only case still left out of consideration gives rise to the following result:

THEOREM. *Let $\boldsymbol{K}$ be a field, complete with respect to a discrete valuation $\boldsymbol{v}$*

of rank r; denote by $0 \subset \boldsymbol{P}_1 \subset \boldsymbol{P}_2 \subset \ldots \subset \boldsymbol{P}_{r-1} \subset \boldsymbol{P}_r = \boldsymbol{M}$ the prime ideals of the valuation ring $\boldsymbol{A}$ of $\boldsymbol{v}$.

Let us give, for every $\boldsymbol{P}_i \neq 0$, an integer $e_i \geqq 1$ so that if $i < j$ then e_i divides e_j. Let us give, for every $\boldsymbol{P}_i \neq 0$, a finite separable extension K_i of $\boldsymbol{K}_i = \boldsymbol{K}/\boldsymbol{v}_{\boldsymbol{P}_i}$, having degree f_i, and assume that for $0 \neq \boldsymbol{P}_i \subset \boldsymbol{P}_j$ we have:

$$f_i = \frac{e_j}{e_i} \cdot f_j. \tag{17}$$

Then, there exists a field K, which is a separable extension of $\boldsymbol{K}$, of finite degree $n = e_r \cdot f_r$ such that if v is the unique extension of $\boldsymbol{v}$ to K then $(v_{P_i}(K) : v_{\boldsymbol{P}_i}(\boldsymbol{K})) = e_i$ and $K/v_{P_i} = K_i$ $(i = 1, \ldots, r)$, where P_i is the only ideal of the valuation ring A of v corresponding to $\boldsymbol{P}_i$.

Proof. The theorem is true for a discrete valuation $\boldsymbol{v}$ of rank 1, as was shown by Krull [5] (1).

We suppose that the theorem has already been proved for a valuation of rank at most $r - 1$.

Let $\boldsymbol{w} = \boldsymbol{v}_{\boldsymbol{P}_1}$, hence $\boldsymbol{K}$ is complete with respect to $\boldsymbol{w}$, and $\bar{\boldsymbol{K}} = \boldsymbol{K}/\boldsymbol{w}$ is complete with respect to $\bar{\boldsymbol{v}} = \boldsymbol{v}/\boldsymbol{P}_1$ (see [7]). Now, $\bar{\boldsymbol{v}}$ is discrete of rank $r-1$, for each prime ideal $\bar{\boldsymbol{P}}_j = \boldsymbol{P}_j/\boldsymbol{P}_1 \neq 0$ in the valuation ring of $\bar{\boldsymbol{v}}$, we have $\bar{\boldsymbol{v}}_{\bar{\boldsymbol{P}}_j} = \boldsymbol{v}_{\boldsymbol{P}_j}/\boldsymbol{P}_1$, $\bar{\boldsymbol{K}}/\bar{\boldsymbol{v}}_{\bar{\boldsymbol{P}}_j} = \boldsymbol{K}/\boldsymbol{v}_{\boldsymbol{P}_j}$, hence, for $\boldsymbol{P}_1 \neq \boldsymbol{P}_i \subset \boldsymbol{P}_j$ we have

$$[K_i : \bar{\boldsymbol{K}}/\bar{\boldsymbol{v}}_{\bar{\boldsymbol{P}}_i}] = \frac{e_j}{e_i} \cdot [K_j : \bar{\boldsymbol{K}}/\bar{\boldsymbol{v}}_{\bar{\boldsymbol{P}}_j}]$$

because of relation (17). Thus, by induction, there exists a field $\bar{K}$, which is a separable extension of $\bar{\boldsymbol{K}}$ of finite degree, $[\bar{K} : \bar{\boldsymbol{K}}] = (e_r/e_1) \cdot f_r$ and such that if $\bar{v}$ is the only extension of $\bar{\boldsymbol{v}}$ to $\bar{K}$ then $(\bar{v}_{\bar{P}_i}(\bar{K}) : \bar{\boldsymbol{v}}_{\bar{\boldsymbol{P}}_i}(\bar{\boldsymbol{K}})) = (e_i/e_1)$, $\bar{K}/\bar{v}_{\bar{P}_i} = K_i$ (for $i = 2, \ldots, r$) where $\bar{P}_i$ is the only prime ideal of the valuation ring $\bar{A}$ of $\bar{v}$ corresponding to $\bar{\boldsymbol{P}}_i$.

Consider now the discrete valuation of rank 1 $\boldsymbol{w}$ of $\boldsymbol{K}$, for which $\boldsymbol{K}$ is complete. Given the integer e_1 and the separable extension $\bar{K} \mid \bar{\boldsymbol{K}}$ of degree $(e_r/e_1) \cdot f_r$, by the validity of the theorem for valuations of rank 1, we deduce the existence of a field K, separable extension of $\boldsymbol{K}$, of degree $e_1 \cdot (e_r/e_1) \cdot f_r = e_r \cdot f_r$ such that if w is the only extension of $\boldsymbol{w}$ to K then $(w(K) : \boldsymbol{w}(\boldsymbol{K})) = e_1$, $K/w = \bar{K}$.

Let v be the unique valuation of K corresponding to $\bar{v}$, that is, finer than w and such that if $w = v_{P_1}$ then $\bar{v} = v/P_1$. Then, v is the only prolongation of $\boldsymbol{v}$ to K, because $\bar{v}$ is the only prolongation of $\bar{\boldsymbol{v}}$ to $\bar{K}$. Moreover $K/v_{Pi} = \bar{K}/\bar{v}_{\bar{P}_i}$ where P_i is the prime ideal of the valuation ring A of v, corresponding to $\boldsymbol{P}_i$, $\bar{P}_i = P_i/P_1$; as $\bar{P}_i$ corresponds to $\boldsymbol{P}_i/\boldsymbol{P}_1 = \bar{\boldsymbol{P}}_i$ then $\bar{P}_i$ extends $\bar{\boldsymbol{P}}_i$ and $\bar{K}/\bar{v}_{\bar{P}_i} = K_i$.

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(1) In case the residue class field $\boldsymbol{K}/\boldsymbol{v} = \bar{\boldsymbol{K}}$ is a *perfect* field, then the theorem might as well be deduced from Hasse-Schmidt-Witt theory of fields, complete with a discrete rank 1 valuation having perfect residue class field (Hasse [3]).

Finally, $(v_{\bar{P}_i}(K) : v_{P_i}(K)) = (\bar{v}_{\bar{P}_i}(\bar{K}) : \bar{v}_{\bar{P}_i}(\bar{K})) \cdot (w(K) : w(K)) = (e_i/e_1) \cdot e_1 = e_i$. This finishes the proof of the theorem.

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